

Linguistic Phenomena in Mathematics

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A Language for Mathematics

- ▶ Joint work with Thomas Barnet-Lamb.
- ▶ Long-term project.
- ▶ Only discuss linguistic aspects here.

Mathematics as a Linguistic Domain

- ▶ Clean, total semantics.
- ▶ Every mathematical term is formally defined.
- ▶ Can extract all syntactic and semantic information from definitions.
- ▶ *Adaptivity* — language starts with a small core, expands via definitions.
- ▶ Benefits of both closed domains and open domains.

Example: Real Mathematics

Sylow's Theorems

Let G be a finite group whose order is divisible by the prime p . Suppose p^m is the highest power of p which is a factor of $|G|$ and set

$$k = \frac{|G|}{p^m}.$$

Then

1. the group G contains at least one subgroup of order p^m ,
2. any two subgroups of G of order p^m are conjugate, and
3. the number of subgroups of G of order p^m is congruent to 1 modulo p and is a factor of k .

Example: MIZAR (truncated)

theorem :: *GROUP_10:12*

for G **being** finite Group,
 p **being** prime (natural number)
holds **ex** P **being** Subgroup of G **st**
 P is_Sylow_ p -subgroup_of_prime p ;

theorem :: *GROUP_10:14*

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Comparison

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  (for P1,P2 being Subgroup of G
    st P1 is_Sylow_p-subgroup_of_prime p
    & P2 is_Sylow_p-subgroup_of_prime p
    holds P1,P2 are_conjugated);
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theorem :: GROUP_10:15
  for G being finite Group,
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  card the_sylow_p-subgroups_of_prime(p,G)
    mod p = 1 &
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    divides ord G;
```

Example: A New Language

Theorem 72 (*“Sylow’s Theorems”*)

Let G be a finite group whose order is divisible by a prime p . Let m be the integer s.t. p^m is the highest power of p which divides $|G|$ and set

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Natural Formal Languages

- ▶ Fuse concepts from formal languages and natural languages.
- ▶ Requires solving substantive some problems, but...
- ▶ Find formal language features counteract weaknesses of natural language and vice versa.
- ▶ E.g. type (FL concept) defangs problems caused by ambiguity (NL concept).
- ▶ Major examples take too much space... .
- ▶ Below: Discuss a minor issue of particular interest to linguists.

Mathematics and Logic

- ▶ Then $V = U \cap H$ for some U in \mathcal{T} , by definition of \mathcal{T}_H , and $U \cap H = i^{-1}(U)$, so $g^{-1}(V) = g^{-1}(i^{-1}(U)) = (i \circ g)^{-1}(U)$.

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Mathematics and Logic

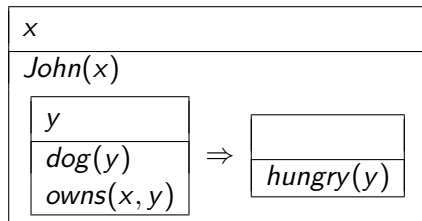
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\uparrow
 unbound variable

Varying the Quantifier

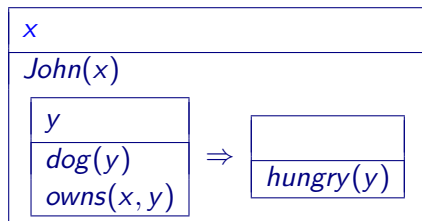
- ▶ $\alpha[U]$ for **some** U in \mathcal{T} and $\beta[U]$.
- ▶ $*\alpha[U]$ for **every** U in \mathcal{T} and $\beta[U]$.
- ▶ Asymmetric treatment of quantifiers required.
- ▶ All standard logics fail.

Discourse Representation Theory



All John's dogs are hungry.
 (I.e. If y is a dog which John owns, then y is hungry.)

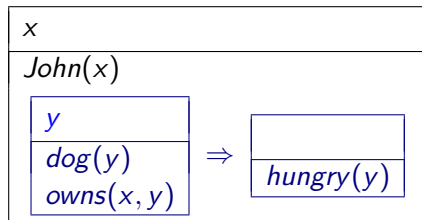
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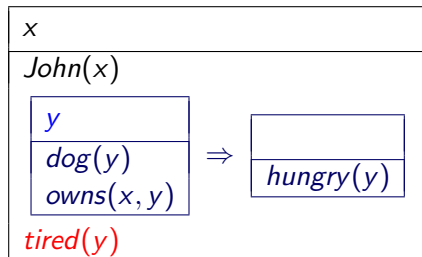
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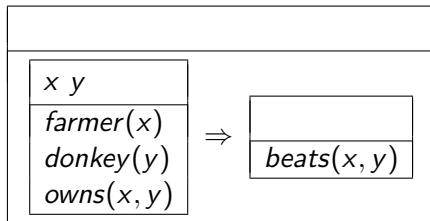
Donkey Sentences and DRT

Every farmer who owns a donkey beats it.

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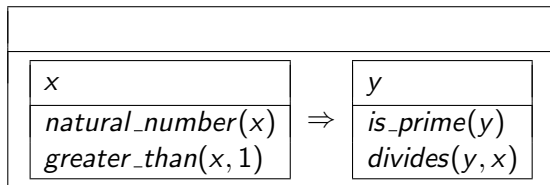
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Mathematical Variables as Referents

Every natural number which is greater than 1 has a prime divisor.

If $n > 1$ is a natural number, then there is a prime p such that $p|n$.

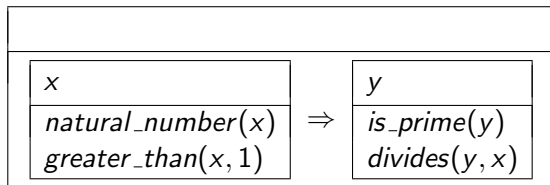
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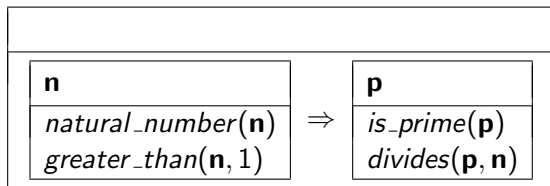
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DRT in Action

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$\alpha[U]$ for some U in \mathcal{T}

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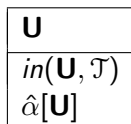
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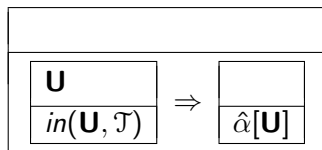
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Conclusion

- ▶ Mathematical language contains variables, normally associated with formal languages.
- ▶ But a semantic theory fitted to formal languages (Predicate Calculus) cannot describe their behaviour.
- ▶ Need to use a theory designed for natural languages (DRT).
- ▶ Such hybridised formal language/natural language concepts recur throughout mathematics.
- ▶ We need a ‘natural formal language’ to describe mathematics.