Identity, Indiscernibility, and Invariance

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1 Indiscernibility and Automorphism

An *automorphism* is an isomorphism from $\mathcal{M} \longrightarrow \mathcal{M}$.
If some automorphism swaps a with b, then $\mathcal{M} \vDash \theta(a) \leftrightarrow \theta(b)$ for every parameter-free $\mathscr{L}$-formula θ (recall week 1). What about the converse?

Example 1. Consider $\mathcal{M}$ with domain $\mathbb{R}$, a single relational symbol $<$, and a constant c_i naming each integer i. Let $a < b$ be such that $\mathcal{M} \vDash \theta(a) \leftrightarrow \theta(b)$. If either a or b is an integer, then $a = b$. Suppose not; there is an integer i such that a, b lie in the open interval $(i, i+1)$. Let π be constant except in the interval $(i, i+1)$, which it *squishes* whilst preserving order, so that it sends a to b. Then π will be an automorphism. □

However, with more complicated $\mathscr{L}$, this can fail.
To generalise this idea, we must move to *elementary extensions* of $\mathcal{M}$.[1]

Lemma 2. (i) If $\mathcal{N} \vDash \text{eldiag}(\mathcal{M})$, there is an $\mathcal{O} \cong \mathcal{N}$ such that $\mathcal{M} \prec \mathcal{O}$.
(ii) If I is a linear order and $\mathcal{M}_i \subseteq \mathcal{M}_j$ whenever $i < j \in I$, then $\bigcup_{i \in I} \mathcal{M}_i$ is an elementary extension of each $\mathcal{M}_i$. □

Definition 3. Let $\mathcal{M}, \mathcal{N}$ be $\mathscr{L}$-structures and $B \subseteq M$. $f : B \longrightarrow N$ is a *partial elementary map* **iff** for all $\mathscr{L}$-formulae φ and all $\overline{b} \in B$, $\mathcal{M} \vDash \varphi(\overline{b}) \Leftrightarrow \mathcal{N} \vDash \varphi(f(\overline{b}))$. □

Lemma 4. Let $\mathcal{M}, \mathcal{N}, \mathscr{L}, B, f$ be as in Definition 3. If $a \in M$ there is $\mathcal{O} \succ \mathcal{N}$ and a partial elementary map $g : B \cup \{a\} \longrightarrow \mathcal{O}$ extending f.

Proof. Let $\Gamma = \{\varphi(v, f(\overline{b})) \mid \mathcal{M} \vDash \varphi(a, \overline{b}) \text{ for some } \overline{b} \in B\}$. If $\Gamma(x) \cup \text{eldiag}(\mathcal{N})$ has a model, then there is a model $\mathcal{O} \succ \mathcal{N}$ with $c \in \mathcal{O}$ witnessing Γ; extending f to g by mapping $a \mapsto c$ will keep our map partial elementary. By Compactness, it suffices to show that every finite subset $\Delta \subseteq \Gamma \cup \text{eldiag}(\mathcal{N})$ is satisfiable. In fact, (the expansion of) $\mathcal{N}$ itself satisfies each such Δ. For it certainly satisfies eldiag$(\mathcal{N})$, and we can treat the (finite) contribution from Γ as some sentence $\varphi(v, f(\overline{b}))$. But $\mathcal{M} \vDash \exists v \varphi(v, \overline{b})$ by hypothesis, and f is partial elementary, so that $\mathcal{N} \vDash \exists v \varphi(v, f(\overline{b}))$, as required. □

Corollary 5. Let $\mathcal{M}, \mathcal{N}, \mathscr{L}, B, f$ be as in Definition 3. Then there is some $\mathcal{O} \succ \mathcal{N}$, and an elementary embedding $g : \mathcal{M} \longrightarrow \mathcal{O}$ extending f.

[1] Proof strategy for remainder of section follows Marker 2002: 117–8.

Proof. Let $\kappa = |M|$, and let $\{a_\alpha \mid \alpha < \kappa\}$ list all the elements in M. Define $\mathcal{N}_0 = \mathcal{N}$, $B_0 = B$, $g_0 = f$ and $B_\alpha = B \cup \{a_\beta \mid \beta < \alpha\}$. We recursively construct an elementary chain thus.

- When $\alpha = \beta + 1$ and $g_\beta : B_\beta \longrightarrow N_\beta$ is partial elementary, we apply Lemma 3, yielding $\mathcal{N}_\alpha \succ \mathcal{N}_\beta$ and $g_\alpha : B_\alpha \longrightarrow N_\alpha$ an elementary map extending g_β.
- When α is a limit ordinal, take $\mathcal{N}_\alpha = \bigcup_{\beta<\alpha} \mathcal{N}_\beta$ and $g_\alpha = \bigcup_{\alpha<\beta} g_\beta$. Certainly, $\mathcal{N}_\alpha \succ \mathcal{N}_\beta$, all $\beta < \alpha$. Moreover, $g_\alpha : B_\alpha \longrightarrow N_\alpha$ is partial elementary.

The limit case is $O = \bigcup_{\alpha<\kappa} \mathcal{N}_\alpha$. Moreover, the map $g_\kappa = \bigcup_{\alpha<\kappa}$ is a partial elementary map $M \longrightarrow O$, i.e. it is an elementary embedding. $\square$

Theorem 6. Suppose $\mathcal{M}$ is an $\mathscr{L}$-structure, with $\bar{a}, \bar{b} \in M^n$. TFAE:

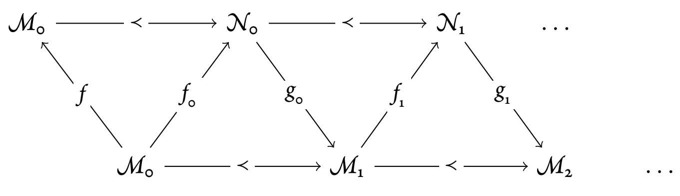
(1) $\mathcal{M} \models \theta(\bar{a}) \leftrightarrow \theta(\bar{b})$, for all parameter-free $\mathscr{L}$-formulae $\theta(\bar{v})$

(2) There is an $\mathcal{N} \succ \mathcal{M}$ with an automorphism π such that $\pi(\bar{a}) = \bar{b}$.

Proof. (2) $\Rightarrow$ (1) is trivial: it follows immediately from the fact that π is an automorphism and $\mathcal{M} \prec \mathcal{N}$ that $\mathcal{M} \models \theta(\bar{a})$ iff $\mathcal{N} \models \theta(\bar{a})$ iff $\mathcal{N} \models \theta(f(\bar{a}))$ iff $\mathcal{N} \models \theta(\bar{b})$.

(1) $\Rightarrow$ (2). Assuming (1), there is a partial elementary map $f : \bar{a} \longrightarrow M$ with $f(\bar{a}) = \bar{b}$. Define $\mathcal{M}_0 := \mathcal{M}$, and use Corollary 5 to create a $\mathcal{N}_0 \succ \mathcal{M}_0$, and an elementary embedding $f_0 : \mathcal{M} \longrightarrow \mathcal{N}_0$ extending f.

- Given $f_i : \mathcal{M}_i \longrightarrow \mathcal{N}_i$ with $M_i \prec N_i$, note that $f_i^{-1} : \mathrm{range}(f_i) \subseteq N_i \longrightarrow \mathcal{M}_i$ is partial elementary, i.e. $\mathcal{N} \models \varphi(\bar{b})$ **iff** $\mathcal{M}_i \models \varphi(f_i^{-1}(\bar{b}))$ for all $b \in \mathrm{range}(f_i)$. Corollary 5 yields an $\mathcal{M}_{i+1} \succ \mathcal{M}_i$ with an elementary embedding $g_i : \mathcal{N}_i \longrightarrow \mathcal{M}_{i+1}$ extending f_i^{-1}, and we can choose $\mathcal{M}_{i+1}$ so that $N_i \prec M_{i+1}$ (this is possible since $M_i \prec N_i$).
- Given $g_i : \mathcal{N}_i \longrightarrow \mathcal{M}_{i+1}$, g_i^{-1} is partial elementary. As above, we can obtain a $\mathcal{N}_{i+1} \succ \mathcal{M}_{i+1}$ with an elementary embedding $f_{i+1} : \mathcal{M}_{i+1} \longrightarrow \mathcal{N}_{i+1}$ extending g_i^{-1}. Since $f_i \subseteq g_i^{-1}$ and $g_i \subseteq g_{i+1}$, we have $f_i \subseteq f_{i+1}$.



In the limit, take $\mathcal{N} = \bigcup_{i<\omega} \mathcal{N}_i = \bigcup_{i<\omega} \mathcal{M}_i$. It is clear that $\mathcal{N} \succ \mathcal{M}$. Define $\pi = \bigcup_{i<\omega} f_i$. It is easy to check that $\pi : \mathcal{N} \longrightarrow \mathcal{N}$ is an elementary map and that π is bijective; so π is an automorphism with $\pi(\bar{a}) = \bar{b}$. $\square$

2 Definability and Invariance

Definition 7. Let p be a set of $\mathscr{L}$-formulas in free variables $v_1, \ldots, v_n$. p is an *n-type* **iff** $p \cup \mathrm{Th}(\mathcal{M})$ is satisfiable. p is a *complete* n-type **iff** $\varphi \in p$ or $\neg\varphi \in p$, for all $\mathscr{L}$-formulae φ with free variables from $v_1, \ldots, v_n$.

$S_n^{\mathcal{M}}$ is the set of all complete n-types.

$\mathrm{tp}^{\mathcal{M}}(\bar{a})$ is the $p \in S_n^{\mathcal{M}}$ realised by $\bar{a}$ in $\mathcal{M}$. □

In these terms, clause (1) of Theorem 6 is that $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{M}}(\bar{b})$.
The set $S_n^{\mathcal{M}}$ has a natural topology on it.

Definition 8. For φ an $\mathscr{L}$ formula, define:

$$[\varphi] := \{p \in S_n^{\mathcal{M}} \mid \varphi \in p\}$$

The *Stone topology* on $S_n^{\mathcal{M}}$ is generated by taking these sets $[\varphi]$ as *basic*. □

Theorem 9. Let $\mathscr{L}$ be a signature not containing R. Let $\mathcal{M}$ be an infinite $\mathscr{L}$-structure and $U \subseteq M^n$. TFAE

(1) $(\mathcal{M}, U) \vDash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow R(\bar{x}))$ for some parameter-free $\mathscr{L}$-formula φ.
(2) For every $(\mathcal{N}, V) \succ (\mathcal{M}, U)$ and every $\mathscr{L}$-automorphism π of $\mathcal{N}$, $\pi(V) = V$.

Proof. (1) ⇒ (2) is trivial again. For the converse, suppose (1) is false. Suppose also (for reductio) that there is no $(\mathcal{M}_0, U_0) \succ (\mathcal{M}, U)$ with n-tuples $\bar{a}, \bar{b} \in M$ such that:

- $(\mathcal{M}_0, U_0) \vDash \theta(\bar{a}) \leftrightarrow \theta(\bar{b})$, for all $\mathscr{L}$-formulae θ
- $\bar{a} \in U_0$ but $\bar{b} \notin U_0$.

For if not, then it would be the case that each type $q \in S_n^{\mathcal{M}}$ uniquely determines whether $R \in q$, when we expand q to be a a type $\in S_n^{(\mathcal{M},U)}$. So take an arbitrary type $p \in [R]$; then $p_{\mathscr{L}} \vDash R(\bar{x})$. By compactness, there is an $\mathscr{L}$-formula γ_p such that $p \in [\gamma_p]$ and $\gamma_p \vDash R(\bar{x})$, so that $[\gamma_p] \subseteq [R]$. So $C = \{[\gamma_p] \mid p \in [R]\}$ is a cover of $[R]$. Since the Stone Space is compact, there is a finite subcover of $[R]$, namely $[\gamma_1], \ldots, [\gamma_n]$.So $[R] = [\gamma_1], \ldots, [\gamma_n]$. Consequently, $(\mathcal{M}, U) \vDash \forall x(R(\bar{x}) \leftrightarrow (\gamma_1(\bar{x}) \vee \ldots \vee \gamma_n(\bar{x})))$. Contradiction.

We now deploy a slight variant of Theorem 6 (see Proposition below) to obtain a model $(\mathcal{N}, V) \succ (\mathcal{M}, U)$ with an $\mathscr{L}$-automorphism π such that $\pi(\bar{a}) = \bar{b}$, $\bar{a} \in V$ and $\bar{b} \notin V$; hence $\pi(V) \neq V$. □

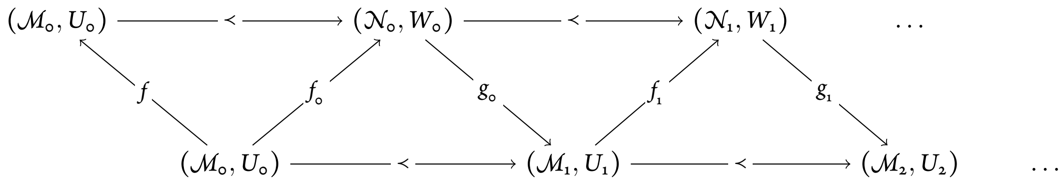
This proof strategy follows Poizat 9.2, but the variant Theorem is proved by deforming Marker's proof of Theorem 6.

Proposition 10. Given $(\mathcal{M}_0, U_0), \overline{a}, \overline{b}$ as in Theorem 9, there is a $(\mathcal{N}, V) \succ (\mathcal{M}_0, U_0)$, with an $\mathcal{L}$-automorphism π such that $\pi(\overline{a}) = \overline{b}$, $\overline{a} \in V$ and $\overline{b} \notin V$.

Proof. We retrace the steps in the proof of Theorem 6, with slight modification. First we prove, exactly as above, a modified version of Corollary 5.

> Let $(\mathcal{M}, U)$, $(\mathcal{N}, V)$ be $\mathcal{L} \cup \{R\}$-structures, $B \subseteq M$ and $f : B \longrightarrow N$ be a partial $\mathcal{L}$-elementary map. There is some $(\mathcal{O}, W) \succ (\mathcal{N}, V)$ and $g : \mathcal{M} \longrightarrow \mathcal{O}$ an $\mathcal{L}$-elementary embedding extending f.

Starting with an $\mathcal{L}$-elementary map $f : \overline{a} \longrightarrow M$ with $f(\overline{a}) = \overline{b}$, we apply this repeatedly, as in Theorem 6, generating:



with each function f_i, g_i an $\mathcal{L}$-elementary embedding, and each $\mathcal{M}_i \prec \mathcal{N}_i \prec \mathcal{M}_{i+1}$. Now define $(N, V) := \bigcup_{i<\omega}(\mathcal{M}_i, U_i)$; obviously $(\mathcal{M}_0, U_0) \prec (\mathcal{N}, V)$. Next, define $\pi := \bigcup_{i<\omega} f_i$. Certainly $\pi(\overline{a}) = \overline{b}$, since $f(\overline{a}) = \overline{b}$. As before, $\pi : N \longrightarrow N$ is an $\mathcal{L}$-automorphism; since π extends f, $\pi(\overline{a}) = \overline{b}$; and since $\overline{a} \in U_0$ and $\overline{b} \notin U_0$ and $(\mathcal{M}_0, U_0) \prec (\mathcal{N}, V)$, $\overline{a} \in V$ and $\overline{b} \notin V$. □

3 Defining Identity

We have considered definability of relations. When is *identity* definable?

Definition 11 (Hilbert–Bernays). For any n+1-ary relation symbol R, say $x \approx_R y$ **iff**

$$\begin{aligned}&\forall z_1 \ldots \forall z_n(Rxz_1z_2 \ldots z_n \leftrightarrow Ryz_1z_2 \ldots z_n) \wedge \\ &\forall z_1 \ldots \forall z_n(Rz_1xz_2 \ldots z_n \leftrightarrow Rz_1yz_2 \ldots z_n) \wedge \ldots \wedge \\ &\forall z_1 \ldots \forall z_n(Rz_1z_2 \ldots z_nx \leftrightarrow Rz_1z_2 \ldots z_ny)\end{aligned}$$

Suppose $\mathscr{L} = \{R_1, \ldots, R_n\}$; then define $x \approx y$ as $\bigwedge_{i \leq n} x \approx_{R_i} y$. □

Theorem 12 (Ketland 3.10). If any formula defines = in $\mathcal{M}$, $x \approx y$ defines =. □

Theorem 13 (Ketland 3.23). If $a \approx b$ then π_{ab} is an automorphism.[2]

Proof. Suppose $a \approx b$. Then for any n+1-ary R:

$$\mathcal{M} \vDash \forall x_1 \ldots x_n(Rx_1 \ldots x_m a x_{m+1} \ldots x_n \leftrightarrow Rx_1 \ldots x_m b x_{m+1} \ldots x_n)$$

So in particular, for all $\overline{d} \in M$:

$$\mathcal{M} \vDash Rd_1 \ldots d_m a d_{m+1} \ldots d_n \leftrightarrow Rd_1 \ldots d_m b d_{m+1} \ldots d_n \tag{1}$$

$$\mathcal{M} \vDash Rd_1 \ldots d_m a d_{m+1} \ldots d_n \leftrightarrow Rd_1 \ldots d_m \pi_{ab}(a) d_{m+1} \ldots d_n \tag{2}$$

$$\mathcal{M} \vDash Rd_1 \ldots d_m a d_{m+1} \ldots d_n \leftrightarrow R\pi_{ab}(d_1) \ldots \pi_{ab}(d_m)\pi_{ab}(a)\pi_{ab}(d_{m+1}) \ldots \pi_{ab}(d_n) \tag{3}$$

Line (1) entails line (2) trivially. Line (2) entails line (3) by the following observation: if $d_i = a$ then $\pi_{ab}(d_i) = b$, and substitution of a for b preserves truth-values since $a \approx b$; similarly if $d_i = b$; and if $d_i \notin \{a, b\}$ then $\pi_{ab}(d_i) = d_i$. □

These results show that (classical) identity is not always definable.
Does this mean that we must take identity as a primitive?
How does it connect with structuralism?

> The position we wish to call *structuralism* holds that: either there may be objects which are not individuals; or at least, if every object in every possible world is an individual, then it is not in all cases due to differences which are purely intrinsic. [Caulton & Butterfield: 40]

Question: Is this an appropriate usage of 'structuralism'?
To answer this, we need to think about a few metaphysical pictures.

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[2] NB: updated since the seminar, with thanks to Adam Caulton, Øystein Linnebo and Jeff Ketland.

4 Metaphysical Theses

Relative to an identity-free relational signature $\mathscr{L}$, we say that a and b are

monadically discernible **iff** $\text{tp}^{\mathcal{M}}(a) \neq \text{tp}^{\mathcal{M}}(b)$

relatively discernible **iff** $\varphi(a, b) \wedge \neg\varphi(b, a)$, for some $\varphi(x, y)$ with only x, y free

weakly discernible **iff** $\varphi(a, b) \wedge \neg\varphi(a, a)$, for some $\varphi(x, y)$ with only x, y free

Monadic $\Rightarrow$ Relative $\Rightarrow$ Weak $\Leftrightarrow$ $a \not\approx b$ (Ketland 3.17–3.20)
We could impose restrictions on φ, but even with this, we have some metaphysical pictures:

SPII. if $a \neq b$, then a is monadically discernible from b.
WPII. if $a \neq b$, then $a \not\approx b$.

If we deny that objects need to be discernible (somehow), then identity is undefinable.
But there are still distinct metaphysical positions left! Caulton & Butterfield offer:

Haec. Each object has its own 'haecceity'. We can model this using distinct *haecceity-symbols*—a monadic predicate N_a for each a—to be treated as a logical constant, so that the extension of N_a is $\{a\}$ in any model. ('Magical names'.)
QII. There are utter indiscernibles, but no such haecceistic properties.

But why do Haec and QII differ? The model theory gives little hint:

- any QII-admissible structure has a unique expansion to a Haec-admissible structure: add haecceity-symbols.
- any Haec-admissible structure has a unique reduction to a QII-admissible structure; ditch the haecceity-symbols.

We need to examine their different attitudes to the question of *what a 'structure' is*.
Let $\mathscr{L}$ contain no haecceity-symbols; let $\mathcal{M}$ be an $\mathscr{L}$-structure; let $\pi : M \longrightarrow M$ be a bijection which is not an automorphism. π induces a structure $\mathcal{P} \cong \mathcal{M}$ and, since π was not an automorphism, $\mathcal{P} \neq \mathcal{M}$ (recall week 1).

Haec. $\mathcal{P}^+$ and $\mathcal{M}^+$ are genuinely *different* structures.
The Haec-admissible structures are (represented by) model-theoretic structures.
QII. This process doesn't really yield a genuinely different **structure**.
The QII-admissible **structures** are (represented by) *isomorphism classes* of model-theoretic structures.
QII invokes a PII for **structures** themselves: PII**S**.

PII**S** *is* distinctively structuralist. Structuralists want to say e.g.: *Zermelo's and von Neumann's ordinal sequences both instantiate the same structure*; the first two objects are isomorphic structures in the model-theoretic sense (recall week 2).
But PII**S** can be combined with WPII, for example.
The question remains: Is '=' a primitive symbol in the structuralist's ideal language?